Solving First Order and First Degree Differential Equations

Bindeshwar Singh Kushwaha

PostNetwork Academy

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$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

Applying Initial Conditions

- **Use** $y = -1$, $x = 0$**:**

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- **Solve the quadratic in y:**

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$

Choosing the Correct Root

- **Two possible solutions:**

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$
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- **Two possible solutions:**

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$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$

- **Apply the initial condition** $y(0) = -1$

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- **Hence, the solution is:**

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- **This is positive for** $x > -2$
- **So, the solution is valid for:**

$$x > -2$$

Definition: Homogeneous Function

- **A real-valued function h of two variables x and y is called a homogeneous function of degree n if:**

$$h(\lambda x, \lambda y) = \lambda^n h(x, y)$$

for all x, y and $\lambda > 0$.

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- **Recognize that the equation is homogeneous of degree 0.**
- **Rewrite as:**

$$\frac{dy}{dx} = 2\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right)$$

Substitution and Reduction

- **Use substitution:** $v = \frac{y}{x} \Rightarrow y = vx$

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- **Rearranged:**

$$\frac{dv}{dx} = \frac{v(2v + 1)}{x}$$

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- **Separate variables:**

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- **Use partial fractions:**

$$\frac{1}{v(v+1)} = \frac{1}{v} - \frac{1}{v+1}$$

- **Now integrate both sides:**

$$\int \left(\frac{1}{v} - \frac{1}{v+1}\right) dv = \int \frac{2}{x}dx$$

Integration and General Solution

- **Left side:**

$$\ln|v| - \ln|v+1|$$

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- **Right side:**

$$2\ln|x| + \ln|c| = \ln|cx^2|$$

- **Combine:**

$$\ln\left|\frac{v}{v+1}\right| = \ln|cx^2| \Rightarrow \frac{v}{v+1} = cx^2$$

Back Substitution

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- **So:**

$$\frac{y/x}{y/x+1} = cx^2 \Rightarrow \frac{y}{y+x} = cx^2$$

- **Rearranged:**

$$\frac{y}{x+y} = cx^2 \quad \textbf{or} \quad y = \frac{cx^3}{1-cx^2}$$

Question : Given Differential Equation

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- **This is a homogeneous differential equation of degree 0.**
- **Use substitution:** $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting $y = vx$

- **Substituting into the equation:**

$$v + x\frac{dv}{dx} = \frac{(vx)^3}{x^3} + \frac{vx}{x}$$

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- **Rearranging:**

$$x\frac{dv}{dx} = v^3 \quad \Rightarrow \quad \frac{1}{v^3}\frac{dv}{dx} = \frac{1}{x}$$

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- **Separate variables and integrate:**

$$\int \frac{1}{v^3}\,dv = \int \frac{1}{x}\,dx$$

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$$\int \frac{1}{v^3}\,dv = \int \frac{1}{x}\,dx$$

- **Result:**

$$-\frac{1}{2v^2} = \ln x + \ln|c|, \quad \textbf{where } c \textbf{ is a constant}$$

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- **Substituting:**

$$-\frac{1}{2\left(\frac{y^2}{x^2}\right)} = \ln x + \ln|c| \Rightarrow y^2 = -\frac{x^2}{2(\ln x + \ln|c|)}$$

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- **Recall:** $v = \frac{y}{x} \Rightarrow v^2 = \frac{y^2}{x^2}$
- **Substituting:**

$$-\frac{1}{2\left(\frac{y^2}{x^2}\right)} = \ln x + \ln|c| \Rightarrow y^2 = -\frac{x^2}{2(\ln x + \ln|c|)}$$

- **Final general solution:**

$$y^2 = -\frac{x^2}{2\ln(x|c|)} \quad \textbf{or} \quad y^2 = -\frac{x^2}{2} \cdot \frac{1}{\ln(x|c|)}$$

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- **Substitute into the equation:**

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$$2v^3y^4[v\,dy + y\,dv] + (v^4y^4 + y^4)\,dy = 0$$
- **Simplify:**
$$2v^3y^4dy + 2v^3y^5dv + (v^4y^4 + y^4)\,dy = 0$$

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- **Simplify:**
$$2v^3y^4dy + 2v^3y^5dv + (v^4y^4 + y^4)\,dy = 0$$
- **Combine like terms:**
$$(2v^3 + v^4 + 1)y^4\,dy + 2v^3y^5\,dv = 0$$

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- **Divide by y:**

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- **Integrate both sides:**

$$\int \frac{2v^3}{3v^4+1}\,dv + \int \frac{1}{y}\,dy = \ln|c|$$

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- **Integrate both sides:**

$$\int \frac{2v^3}{3v^4+1}\,dv + \int \frac{1}{y}\,dy = \ln|c|$$

- **First integral:**

$$\frac{1}{6}\ln(3v^4+1) + \ln|y| = \ln|c|$$

Back Substitution and Final Answer

- **Combine logs:**

$$\ln[(3v^4 + 1)^{1/6} \cdot y] = \ln|c| \Rightarrow (3v^4 + 1)^{1/6} \cdot y = c$$

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- **Raise both sides to power 6:**
$$(3v^4+1)y^6 = c_1 \quad \textbf{where } c_1 = c^6$$

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$$(3v^4+1)y^6 = c_1 \quad \textbf{where } c_1 = c^6$$
- **Recall** $v = \frac{x}{y} \Rightarrow v^4 = \frac{x^4}{y^4}$

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- **Recall** $v = \frac{x}{y} \Rightarrow v^4 = \frac{x^4}{y^4}$
- **Final form:**
$$3x^4y^2 + y^6 = c_1$$

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Thank You!