Linear Combinations and Spanning Sets

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- **A vector $v \in V$ is a linear combination of vectors $u_1, u_2, \ldots, u_m$ in V if there exist scalars $a_1, a_2, \ldots, a_m \in K$ such that:**

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m$$

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$$v = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m$$

- **Alternatively, v is a linear combination of $u_1, u_2, \ldots, u_m$ if there is a solution to:**

$$v = x_1 u_1 + x_2 u_2 + \cdots + x_m u_m$$

where $x_1, x_2, \ldots, x_m$ are unknown scalars.

Example : Linear Combination in $\mathbb{R}^3$

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- $u_2 = (2, 3, 7)$
- $u_3 = (3, 5, 6)$
- **Seek scalars** x, y, z **such that** $v = xu_1 + yu_2 + zu_3$

Matrix Equation Form

$$\begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} + z \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

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$$\textbf{or the system: } \begin{cases} x + 2y + 3z = 3 \\ 2x + 3y + 5z = 7 \\ 3x + 7y + 6z = -4 \end{cases}$$

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- **Reduce to echelon form:**

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- **So,** $v = 2u_1 - 4u_2 + 3u_3$

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- **We seek scalars** x, y, z **such that** $v = xp_1 + yp_2 + zp_3$
- **i.e.,**

$$3t^2 + 5t - 5 = x(t^2 + 2t + 1) + y(2t^2 + 5t + 4) + z(t^2 + 3t + 6)$$

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- **Equating coefficients with $3t^2 + 5t - 5$ gives the system:**

$$\begin{cases} x + 2y + z = 3 \\ 2x + 5y + 3z = 5 \\ x + 4y + 6z = -5 \end{cases}$$

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- **Thus,**

$$v = 3p_1 + p_2 - 2p_3$$

Alternate Method: Coefficients of Identity

- **Use the identity for all t:**

$$x(t^2 + 2t + 1) + y(2t^2 + 5t + 4) + z(t^2 + 3t + 6) = 3t^2 + 5t - 5$$

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- **Solve this system and get:**

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- **Solve this system and get:**

$$x = 3, \quad y = 1, \quad z = -2$$

- **Again,** $v = 3p_1 + p_2 - 2p_3$

Spanning Sets

Let V be a vector space over a field K. Vectors $u_1, u_2, \ldots, u_m$ in V are said to span V if every vector in V is a linear combination of them:

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Remark 3: If one $u_k = 0$, then removing it still spans V.

Example – Part (a)

Let $V = \mathbb{R}^3$.

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- **Any vector** $v = (a, b, c)$ **in** $\mathbb{R}^3$**:**

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- **Any vector** $v = (a, b, c)$ **in** $\mathbb{R}^3$**:**
$$v = ae_1 + be_2 + ce_3$$
- **Example:** $(5, -6, 2) = -5e_1 - 6e_2 + 2e_3$

Example – Part (b)

Claim: $\{w_1, w_2, w_3\}$ **also spans** $\mathbb{R}^3$

- **Vectors:**

$$w_1 = (1, 1, 1), \quad w_2 = (1, 1, 0), \quad w_3 = (1, 0, 0)$$

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$$w_1 = (1,1,1), \quad w_2 = (1,1,0), \quad w_3 = (1,0,0)$$
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- **Example:** $v = (5, -6, 2) = 2w_1 - 8w_2 + 11w_3$

Example – Part (c)

Claim: $\{u_1, u_2, u_3\}$ **does not span** $\mathbb{R}^3$

- **Vectors:**

$$u_1 = (1, 2, 3), \quad u_2 = (1, 5, 3), \quad u_3 = (1, 5, 9)$$

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- **Vectors:**
$$u_1 = (1, 2, 3), \quad u_2 = (1, 5, 3), \quad u_3 = (1, 5, 9)$$
- **Counter example:** $v = (2, 7, 8)$ **cannot be written as a linear combination of** u_i

Example – Spanning Set for $P_n(t)$

Vector space: $V = P_n(t)$ **(all polynomials of degree** $\leq n$**)**

- **(a) Any polynomial in** $P_n(t)$ **can be written as a linear combination of:**

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- **This shows another valid spanning set for** $P_n(t)$**.**

Example – Spanning Set for $M_{2,2}$

Vector space: $M = M_{2,2}$ **(all** 2×2 **matrices)**

- **Consider the matrices:**

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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- **Any** 2×2 **matrix can be expressed as a linear combination of these.**
- **For example:**

$$A = \begin{bmatrix} 5 & -6 \\ 7 & 8 \end{bmatrix} = 5E_{11} - 6E_{12} + 7E_{21} + 8E_{22}$$

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$$A = \begin{bmatrix} 5 & -6 \\ 7 & 8 \end{bmatrix} = 5E_{11} - 6E_{12} + 7E_{21} + 8E_{22}$$

- **Therefore,** $E_{11}, E_{12}, E_{21}, E_{22}$ **span** $M_{2,2}$**.**

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