Moments of Binomial Distribution

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Moment Definition

Let $X \sim B(n, p)$ be a binomial random variable.

- The r^{th} order moment about origin:

$$\mu_r' = \mathbb{E}(X^r) = \sum_{x=0}^{n} x^r \cdot \mathbb{P}(X = x)$$

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- Use binomial PMF: $\mathbb{P}(X = x) = \binom{n}{x} p^x q^{n-x}$, where $q = 1 - p$

Substitute Binomial PMF

- **Substituting into the formula:**

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- **Use identity:** $x \cdot \binom{n}{x} = n \cdot \binom{n-1}{x-1}$

$$\mu_1' = \sum_{x=1}^{n} n \cdot \binom{n-1}{x-1} p^x q^{n-x}$$

Index Substitution

- **Factor out constants:**

$$\mu_1' = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x}$$

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$$\mu_1' = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x}$$

- **Let** $r = x - 1 \Rightarrow x = r + 1$
- **Change limits:** $x = 1 \Rightarrow r = 0,\ x = n \Rightarrow r = n - 1$

$$\mu_1' = np \sum_{r=0}^{n-1} \binom{n-1}{r} p^{r} q^{n-1-r}$$

Final Result and Mean

- **Recognize binomial expansion:**

$$\sum_{r=0}^{n-1} \binom{n-1}{r} p^r q^{n-1-r} = (p+q)^{n-1}$$

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- **Therefore:**

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- **Mean of the Binomial Distribution:**

$$\mu = \mu_1' = np$$

Second Moment: $\mu_2' = \mathbb{E}(X^2)$

- **Start with:**

$$\mu_2' = \mathbb{E}(X^2) = \sum_{x=0}^{n} x^2\, \mathbb{P}(X = x)$$

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- **Hence:**

$$\mu_2' = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$$

- **Write** $x^2 = x(x-1) + x$

$$\mu_2' = \sum_{x=0}^{n} [x(x-1) + x] \binom{n}{x} p^x q^{n-x}$$

$x(x-1) + x$

$x^2 - x + x$

Split the Summation

- **Split into two parts:**

$$\mu_2' = \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x} + \sum_{x=0}^{n} x\binom{n}{x} p^x q^{n-x}$$

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- **So,**

$$\mu_2' = n(n-1)\sum_{x=2}^{n} \binom{n-2}{x-2} p^x q^{n-x} + \mu_1'$$

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- **Change index: let** $r = x - 2 \Rightarrow x = r + 2$

Index Change and Simplification

- **The sum becomes:**

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- **So,**

$$\mu_2' = n(n-1)p^2 + \mu_1' = n(n-1)p^2 + np$$

Variance of Binomial Distribution

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- **Substituting:**

$$\mathbf{Var}(X) = n(n-1)p^2 + np - (np)^2$$

- **Simplify:**

$$= n(n-1)p^2 + np - n^2p^2 = np(1-p) = npq$$

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- **Write** $x^3 = x(x-1)(x-2) + 3x(x-1) + x$

$$\mu_3' = \sum_{x=0}^{n} [x(x-1)(x-2) + 3x(x-1) + x] \binom{n}{x} p^x q^{n-x}$$

Split into Three Summations

- **Break into three sums:**

$$\mu_3' = \sum x(x-1)(x-2)\binom{n}{x}p^x q^{n-x} + 3\sum x(x-1)\binom{n}{x}p^x q^{n-x} + \sum x\binom{n}{x}p^x q^{n-x}$$

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- **Use identities:**

$$x(x-1)(x-2)\binom{n}{x} = n(n-1)(n-2)\binom{n-3}{x-3}$$

$$x(x-1)\binom{n}{x} = n(n-1)\binom{n-2}{x-2}$$

$$x\binom{n}{x} = n\binom{n-1}{x-1}$$

Substitute and Simplify

- **First sum becomes:**

$$n(n-1)(n-2)p^3 \sum_{x=3}^{n} \binom{n-3}{x-3} p^{x-3} q^{n-x} = n(n-1)(n-2)p^3$$

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$$3n(n-1)p^2$$

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$$np$$

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$$3n(n-1)p^2$$

- **Third sum:**

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- **So,**

$$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

Central Moment and Skewness

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$$\mu_3 = \mu_3' - 3\mu_2\mu_1' + 2(\mu_1')^3$$

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- **Third central moment:**
$$\mu_3 = \mu_3' - 3\mu_2\mu_1' + 2(\mu_1')^3$$
- **Substituting:**
$$\mu_3 = npq(1 - 2p)$$

Fourth Raw Moment: $\mu'_4 = \mathbb{E}[X^4]$

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- **Expand x^4 as:**

$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$

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- **Substitute into summation:**

$$\mu'_4 = \mu_4 + 6\mu_3 + 7\mu_2 + \mu_1$$

Substitute in Terms of n, p

- **From binomial properties, we know:**

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- **Simplified central moment:**

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1 + 6\mu_2'(\mu_1)^2 - 3(\mu_1)^4$$

- **Skewness:**

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \left[\frac{npq(q-p)}{(npq)^{3/2}}\right]^2 = \frac{(q-p)^2}{npq},$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{q-p}{\sqrt{npq}}$$

Skewness and Kurtosis for Binomial

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$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq}$$

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- **Kurtosis:**

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq}$$

- **Excess Kurtosis:**

$$\gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq}$$

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- $\mu_4 = npq[1 + 3(n - 2)pq]$
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- **Kurtosis:**

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Thank You!