# Vector and Vector Space in Linear Algebra

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# Definition of $\mathbb{R}^n$

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- **The term scalar is used for elements of $\mathbb{R}$.**

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- **Example: $(1, 2, 3) \neq (2, 3, 1)$ even though they contain the same numbers.**
- **The vector $(0, 0, \ldots, 0)$ is called the zero vector, denoted by $0$.**

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- **The first two belong to** $\mathbb{R}^2$**, the last two to** $\mathbb{R}^3$**.**

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- **Scalar Multiplication: Assigns to any $u \in V$, $k \in K$ a product $ku \in V$.**

**Then $V$ is a vector space over the field $K$ if the following axioms hold for all vectors $u, v, w \in V$:**

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- **[M4]** $1u = u$, **for the unit scalar** $1 \in K$

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- **Vector sum** $v_1 + v_2 + \cdots + v_n$ **is associative.**
- **Zero vector is unique, and each vector has a unique negative.**
- **If** $u + w = v + w$, **then** $u = v$ **(Cancellation Law).**
- **Subtraction:** $u - v = u + (-v)$

# Vector Space: $K^n$

**Let $K$ be a field. Then $K^n$ is the set of all $n$-tuples of elements in $K$:**

$$K^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in K\}$$

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- **We regard $K^n$ as a vector space over $K$.**

# Polynomial Space $P(\mathbb{F})$

**Let $P(\mathbb{F})$ be the set of all polynomials:**

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

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- $P(\mathbb{F})$ **is a vector space over** $\mathbb{F}$

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# Polynomial Subspace $P_n(\mathbb{F})$

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- **$P_n(\mathbb{F})$ is a subspace of $P(\mathbb{F})$**
- **Closed under addition and scalar multiplication**

# Matrix Space $M_{mn}(\mathbb{F})$

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# Function Space $F(X)$

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