Vector Subspace in Linear Algebra

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Note: If $W \subseteq V$ and satisfies the vector space axioms, then it is a subspace. We use a simplified criterion below.

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Equivalently: For any $u, v \in W$ and scalars $a, b \in K$,
$au + bv \in W$

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- **Every vector space V has two subspaces:**
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- **These are called the trivial subspaces of V**

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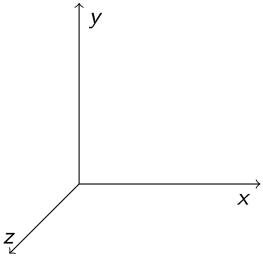

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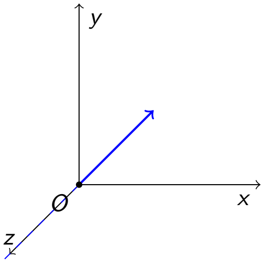

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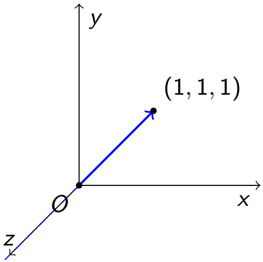

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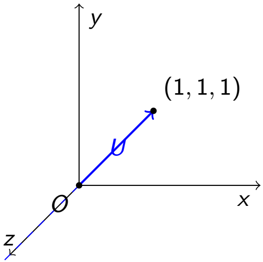

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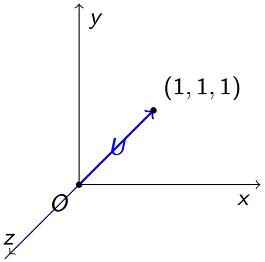

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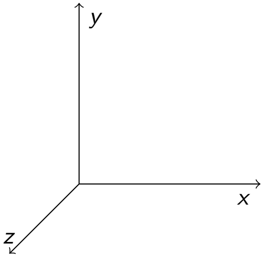

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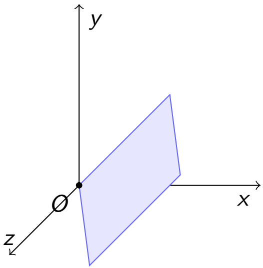

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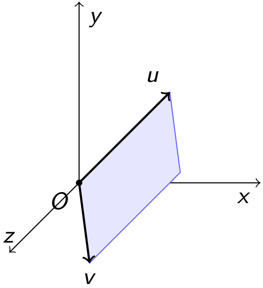

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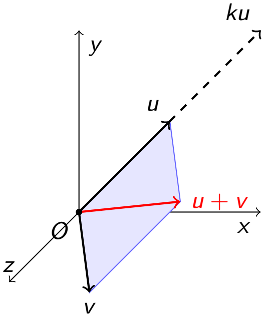

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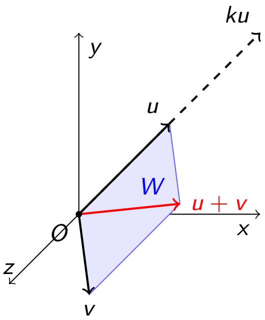

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Let $V = M_{n,n}$, the vector space of $n \times n$ matrices.

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- **Similarly, W_2 is also a subspace of V**

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- **Example polynomials in $Q(t)$:**

$$p_1 = 3 + 4t^2 - 5t^6$$
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Thank You!