Linear Dependence and Independence

Post Network Academy

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- Otherwise, the vectors are **linearly independent**.

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- If this is the only solution, the vectors are linearly independent.
- If a nonzero solution exists, the vectors are linearly dependent.

Set of Vectors

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- An infinite set is linearly dependent if there exist vectors that are linearly dependent within it.

Linear Dependence in $\mathbb{R}^3$

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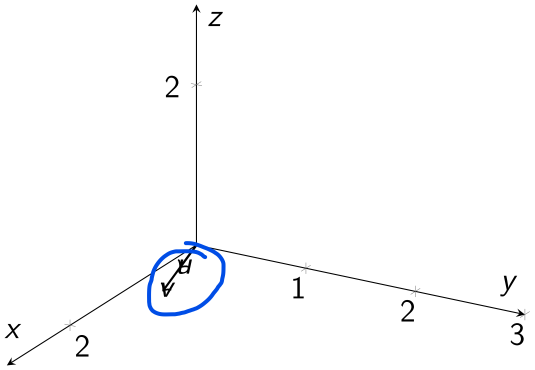
- (b) Any three vectors u, v, w in $\mathbb{R}^3$ are linearly dependent if and only if they lie on the same plane through the origin.
- Any four or more vectors in $\mathbb{R}^3$ are automatically linearly dependent.

Example: Two Vectors on the Same Line



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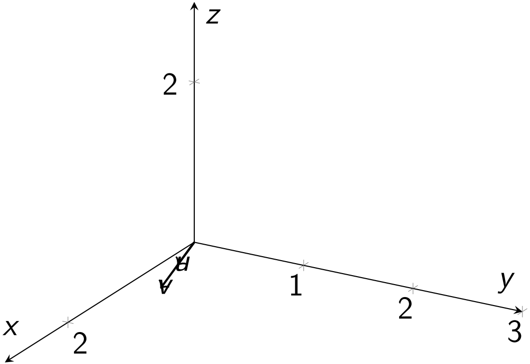
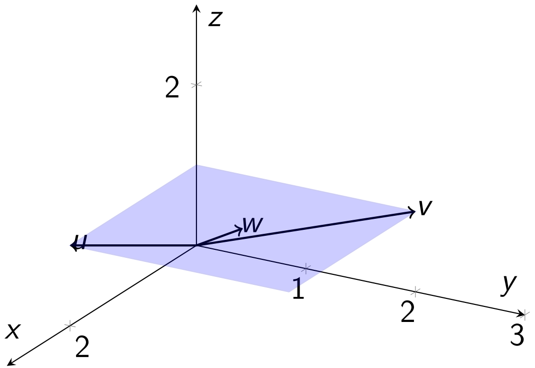


Observation: Both vectors lie on the same line through the origin ⇒ they are linearly dependent.

Example: Three Vectors on the Same Plane



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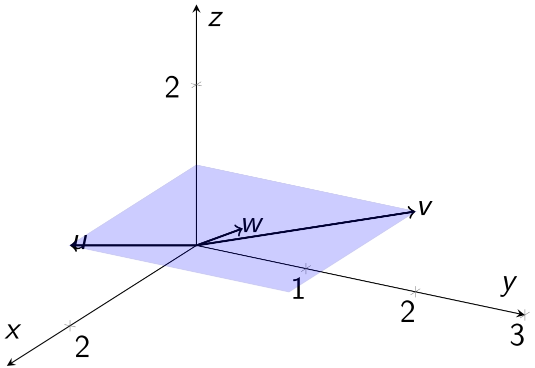

Observation: All three vectors lie on the same plane through the origin ⇒ they are linearly dependent.

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- Non-trivial solution exists $\Rightarrow$ Linearly Dependent.

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- Only trivial solution $\Rightarrow$ Linearly Independent.

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- Substitute $t = \pi$: $\quad x(0) + y(e^\pi) + z(\pi^2) = 0 \quad \Rightarrow \quad z = 0$

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- Only trivial solution $\Rightarrow$ Linearly Independent.

Remarks with Numerical Examples

- **Remark 1:** Suppose 0 is one of the vectors $v_1, v_2, \ldots, v_m$, say $v_1 = 0$. Then the vectors must be linearly dependent.

(1, 2, 0), (2, 3, 1) (0, 0, 0) ✓

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$(2,3,4) \in R^3$

$k(2,3,4) = 0$

$k = 0$

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$u = (2, 4, 5)$ $v = (4, 8, 10)$

$v = 2u$

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