Geometric Distribution Made Simple — Stepwise Approach

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12.2 Geometric Distribution

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- Trials are independent, and we continue performing them until the first success occurs.
- Let X be the number of failures preceding the first success.
- Then X can take values $0, 1, 2, 3, \ldots$

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- And so on...

General Form

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- The respective probabilities $p, qp, q^2p, q^3p, \ldots$ form a geometric progression with common ratio q.
- Hence, this probability distribution is known as the **Geometric Distribution**.

Definition

- A random variable X is said to follow a geometric distribution if it assumes non-negative integer values and its probability mass function is

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- The sum of all probabilities is

$$\sum_{x=0}^{\infty} q^x p = p(1 + q + q^2 + q^3 + \ldots) = 1.$$

Example: Geometric Distribution – Die Problem

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- An unbiased die is cast until 6 appears.
- Find the probability that it must be cast more than five times.

Step 1: Define Success and Failure

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- Probability of failure:

$$q = 1 - p = \frac{5}{6}$$

Step 2: Define Random Variable

- Let $\boldsymbol{X}$ = number of failures before first success.

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- Let X = number of failures before first success.
- Then by geometric distribution:

$$P(X = x) = q^x p, \quad x = 0, 1, 2, \ldots$$

Step 3: Desired Probability

- We want:

$$\boldsymbol{P}\textbf{(}\text{Die is cast more than 5 times}\textbf{)}$$

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$$P(\text{Die is cast more than 5 times})$$

- i.e.

$$P(X \geq 5)$$

Step 4: Express as a Series

- $$P(X \geq 5) = P(X = 5) + P(X = 6) + P(X = 7) + \dots$$

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- $$P(X \geq 5) = P(X = 5) + P(X = 6) + P(X = 7) + \dots$$
- Substitute:
$$= \left(\frac{5}{6}\right)^5 \left(\frac{1}{6}\right) \left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \dots\right]$$

Step 5: Simplify Using Geometric Series

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- Here $r = \frac{5}{6}$.
- Hence,

$$= \left(\frac{5}{6}\right)^5 \left(\frac{1}{6}\right) \left(\frac{1}{1-\frac{5}{6}}\right)$$

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- Hence, the probability that the die must be cast more than five times is:

$$\boxed{\left(\frac{5}{6}\right)^5}$$

Mean of the Geometric Distribution

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- Shifting the index for simplification:

$$= p \sum_{x=1}^{\infty} xq^{x-1}$$

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- Since:

$$\frac{d}{dq}(q^x) = xq^{x-1}$$

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- The infinite geometric series is:

$$\sum_{x=1}^{\infty} q^x = \frac{q}{1-q}$$

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- Simplifying:

$$= \frac{1-q+q}{(1-q)^2} = \frac{1}{(1-q)^2}$$

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- Hence,

$$\boxed{E(X) = \frac{q}{p}}$$

Variance of Geometric Distribution

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- Recall that for geometric distribution:

$$P(X = x) = q^x p, \quad x = 0, 1, 2, \ldots$$

- Therefore,

$$E(X^2) = \sum_{x=0}^{\infty} x^2 q^x p$$

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- We need a similar trick using differentiation.

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- Differentiate again:

$$\sum_{x=1}^{\infty} x(x-1) q^{x-2} = \frac{2}{(1-q)^3}$$

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Forming $\sum x^2 q^{x-1}$

- We know:
$$x^2 = x(x-1) + x$$
- Therefore,
$$\sum x^2 q^{x-1} = \sum x(x-1)q^{x-1} + \sum x q^{x-1}$$
- Multiply both sides by q where required and simplify using derivatives.

Simplifying the Series

- Using the differentiation results:

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- and

$$\sum_{x=1}^{\infty} xq^{x-1} = \frac{1}{(1-q)^2}$$

- Hence,

$$\sum x^2 q^{x-1} = \frac{2q}{(1-q)^3} + \frac{1}{(1-q)^2}$$

Compute $E(X^2)$ and $V(X)$

- Multiply by p to get:

$$E(X^2) = p\left(\frac{2q}{(1-q)^3} + \frac{1}{(1-q)^2}\right)$$

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$$E(X^2) = \frac{q(2-q)}{p^2}$$

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- Simplify using $1 - q = p$:

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- Finally,

$$V(X) = E(X^2) - [E(X)]^2$$

Final Result

- Substitute:

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- Hence,

$$V(X) = \frac{q(2-q)}{p^2} - \left(\frac{q}{p}\right)^2$$

- Simplifying gives:

$$\boxed{V(X) = \frac{q}{p^2}}$$

Remark 1: Geometric Distribution

Variance and Mean Relationship

- Variance $= \dfrac{q}{p^2} = \dfrac{q}{p \cdot p} = \dfrac{\text{Mean}}{p}$

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- $\text{Variance} = \dfrac{q}{p^2} = \dfrac{q}{p \cdot p} = \dfrac{\text{Mean}}{p}$
- $\Rightarrow \text{Variance} > \text{Mean}$ $\left[\text{Since } p < 1 \Rightarrow \dfrac{\text{Mean}}{p} > \text{Mean}\right]$

Remark 1: Geometric Distribution

Variance and Mean Relationship

- Variance $= \dfrac{q}{p^2} = \dfrac{q}{p \cdot p} = \dfrac{\text{Mean}}{p}$
- $\Rightarrow$ Variance $>$ Mean [Since $p < 1 \Rightarrow \dfrac{\text{Mean}}{p} > \text{Mean}$]
- Hence, unlike the binomial distribution, variance of the geometric distribution is greater than the mean.

Example 2: Comment on the Following

Given Statement

- The mean and variance of a geometric distribution are 4 and 3 respectively.

Solution (Part 1)

Assuming Parameter p

- Let p be the probability of success in each trial.

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- Let p be the probability of success in each trial.
- Then, Mean $= \dfrac{q}{p} = 4$ and Variance $= \dfrac{q}{p^2} = 3$

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Derivation

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- $\Rightarrow \dfrac{1}{p} = \dfrac{3}{4}$
- $\Rightarrow p = \dfrac{4}{3}$, which is impossible since probability can never exceed unity.
- Hence, the given statement is wrong.

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Thank You!