Negative Binomial Distribution

A Detailed Step-by-Step Explanation

Bindeshwar Singh Kushwaha
PostNetwork Academy

Introduction: Relation with Geometric Distribution

- The negative binomial distribution is a generalisation of the geometric distribution.

Introduction: Relation with Geometric Distribution

- The negative binomial distribution is a generalisation of the geometric distribution.
- It describes the number of failures before the ***r***-th success in a sequence of Bernoulli trials.

Introduction: Relation with Geometric Distribution

- The negative binomial distribution is a generalisation of the geometric distribution.
- It describes the number of failures before the r-th success in a sequence of Bernoulli trials.
- When $r = 1$, it reduces to the geometric distribution.

Definition of Random Variable

- Let $\boldsymbol{X}$ denote the number of failures before the $\boldsymbol{r}$-th success.

Definition of Random Variable

- Let $\boldsymbol{X}$ denote the number of failures before the $\boldsymbol{r}$-th success.
- Each trial has:

Definition of Random Variable

- Let $\boldsymbol{X}$ denote the number of failures before the $\boldsymbol{r}$-th success.
- Each trial has:
  - Probability of success = $\boldsymbol{p}$

Definition of Random Variable

- Let $\boldsymbol{X}$ denote the number of failures before the $\boldsymbol{r}$-th success.
- Each trial has:
  - Probability of success = $\boldsymbol{p}$
  - Probability of failure = $\boldsymbol{(1-p)}$

Definition of Random Variable

- Let $\boldsymbol{X}$ denote the number of failures before the $\boldsymbol{r}$-th success.
- Each trial has:
  - Probability of success = $\boldsymbol{p}$
  - Probability of failure = $\boldsymbol{(1-p)}$
- Then, $\boldsymbol{X}$ takes values $\boldsymbol{0, 1, 2, \ldots}$

Definition of Random Variable

- Let X denote the number of failures before the r-th success.
- Each trial has:
  - Probability of success = p
  - Probability of failure = $(1 - p)$
- Then, X takes values $0, 1, 2, \ldots$
- We are interested in $P(X = x)$, the probability of observing x failures before the r-th success.

Derivation of Probability Mass Function (PMF)

- To have x failures before the r-th success:

Derivation of Probability Mass Function (PMF)

- To have x failures before the r-th success:
  - The last (i.e. $(x + r)$-th) trial must be a success.

Derivation of Probability Mass Function (PMF)

- To have x failures before the r-th success:
  - The last (i.e. $(x + r)$-th) trial must be a success.
  - The previous $(x + r - 1)$ trials must contain $(r - 1)$ successes and x failures.

Derivation of Probability Mass Function (PMF)

- To have x failures before the r-th success:
  - The last (i.e. $(x + r)$-th) trial must be a success.
  - The previous $(x + r - 1)$ trials must contain $(r - 1)$ successes and x failures.
- Number of ways to arrange $(r - 1)$ successes in $(x + r - 1)$ trials:

$$\binom{x + r - 1}{r - 1}$$

Derivation of Probability Mass Function (PMF)

- To have x failures before the r-th success:
  - The last (i.e. $(x+r)$-th) trial must be a success.
  - The previous $(x+r-1)$ trials must contain $(r-1)$ successes and x failures.
- Number of ways to arrange $(r-1)$ successes in $(x+r-1)$ trials:

$$\binom{x+r-1}{r-1}$$

- Hence,

$$P(X=x) = \binom{x+r-1}{r-1}(1-p)^x p^r$$

Probability Mass Function (PMF)

PMF of Negative Binomial Distribution

$$P(X = x) = \binom{x + r - 1}{r - 1}(1 - p)^x p^r, \quad x = 0, 1, 2, \ldots$$

- p — probability of success in a single trial.

Probability Mass Function (PMF)

PMF of Negative Binomial Distribution

$$P(X = x) = \binom{x + r - 1}{r - 1}(1 - p)^x p^r, \quad x = 0, 1, 2, \ldots$$

- p — probability of success in a single trial.
- $(1 - p)$ — probability of failure.

Probability Mass Function (PMF)

PMF of Negative Binomial Distribution

$$P(X = x) = \binom{x + r - 1}{r - 1}(1 - p)^x p^r, \quad x = 0, 1, 2, \ldots$$

- p — probability of success in a single trial.
- $(1 - p)$ — probability of failure.
- r — number of required successes.

Probability Mass Function (PMF)

PMF of Negative Binomial Distribution

$$P(X = x) = \binom{x + r - 1}{r - 1}(1 - p)^x p^r, \quad x = 0, 1, 2, \ldots$$

- p — probability of success in a single trial.
- $(1 - p)$ — probability of failure.
- r — number of required successes.
- $\binom{x+r-1}{r-1}$ — number of ways to arrange $(r - 1)$ successes among $(x + r - 1)$ trials.

Mean and Variance: Binomial Distribution

- The mean and variance of the negative binomial distribution can be understood by comparing it with the binomial distribution.

Mean and Variance: Binomial Distribution

- The mean and variance of the negative binomial distribution can be understood by comparing it with the binomial distribution.
- For the binomial distribution, the probabilities for $X = 0, 1, 2, \ldots$ are the successive terms of the binomial expansion of $(q + p)^n$.

Mean and Variance: Binomial Distribution

- The mean and variance of the negative binomial distribution can be understood by comparing it with the binomial distribution.
- For the binomial distribution, the probabilities for $X = 0, 1, 2, \ldots$ are the successive terms of the binomial expansion of $(q + p)^n$.
- The mean and variance are:
  - **Mean:** Mean $= np = (n)(p)$

Mean and Variance: Binomial Distribution

- The mean and variance of the negative binomial distribution can be understood by comparing it with the binomial distribution.
- For the binomial distribution, the probabilities for $X = 0, 1, 2, \ldots$ are the successive terms of the binomial expansion of $(q + p)^n$.
- The mean and variance are:
  - **Mean:** Mean $= np = (n)(p)$
  - **Variance:** Var $= npq = (n)(p)(q)$

Mean and Variance: Binomial Distribution

- The mean and variance of the negative binomial distribution can be understood by comparing it with the binomial distribution.
- For the binomial distribution, the probabilities for $X = 0, 1, 2, \ldots$ are the successive terms of the binomial expansion of $(q + p)^n$.
- The mean and variance are:
  - **Mean:** Mean $= np = (n)(p)$
  - **Variance:** Var $= npq = (n)(p)(q)$
- This represents the product of:
  - the index n
  - the second term in $(q + p)$

Mean and Variance: Binomial Distribution

- The mean and variance of the negative binomial distribution can be understood by comparing it with the binomial distribution.
- For the binomial distribution, the probabilities for $X = 0, 1, 2, \ldots$ are the successive terms of the binomial expansion of $(q + p)^n$.
- The mean and variance are:
  - **Mean:** Mean $= np = (n)(p)$
  - **Variance:** Var $= npq = (n)(p)(q)$
- This represents the product of:
  - the index n
  - the second term in $(q + p)$

Mean and Variance: Negative Binomial (Mean)

- For the negative binomial distribution ($X = 0, 1, 2, \ldots$), the probabilities are successive terms of:

$$\left[\frac{1}{p} + \left(\frac{q}{p}\right)\right]^r$$

Mean and Variance: Negative Binomial (Mean)

- For the negative binomial distribution ($X = 0, 1, 2, \ldots$), the probabilities are successive terms of:

$$\left[\frac{1}{p} + \left(\frac{q}{p}\right)\right]^r$$

- Thus, the mean is:

$$\text{Mean} = (\text{index}) \times (\text{second term}) = (-r)\left(-\frac{q}{p}\right) = \frac{rq}{p}$$

Mean and Variance: Negative Binomial (Mean)

- For the negative binomial distribution ($X = 0, 1, 2, \ldots$), the probabilities are successive terms of:

$$\left[\frac{1}{p} + \left(\frac{q}{p}\right)\right]^r$$

- Thus, the mean is:

$$\text{Mean} = (\text{index}) \times (\text{second term}) = (-r)\left(-\frac{q}{p}\right) = \frac{rq}{p}$$

Mean and Variance: Negative Binomial (Variance)

- The variance is:

$$\text{Var} = (-r)\left(-\frac{q}{p}\right)\left(\frac{1+\frac{q}{p}}{p}\right)$$

Mean and Variance: Negative Binomial (Variance)

- The variance is:

$$\text{Var} = (-r)\left(-\frac{q}{p}\right)\left(\frac{1+\frac{q}{p}}{p}\right)$$

- Simplifying:

$$\text{Var} = \frac{rq}{p^2}$$

Mean and Variance: Negative Binomial (Variance)

- The variance is:

$$\text{Var} = (-r)\left(-\frac{q}{p}\right)\left(\frac{1+\frac{q}{p}}{p}\right)$$

- Simplifying:

$$\text{Var} = \frac{rq}{p^2}$$

Example 1: Setup

- **Problem:** Find the probability that the third head turns up in 5 tosses of an unbiased coin.

Example 1: Setup

- **Problem:** Find the probability that the third head turns up in 5 tosses of an unbiased coin.
- This is a **negative binomial distribution** problem.

Example 1: Setup

- **Problem:** Find the probability that the third head turns up in 5 tosses of an unbiased coin.
- This is a **negative binomial distribution** problem.
- Given:
  - $\boldsymbol{p = \frac{1}{2}}$ (success = head)

Example 1: Setup

- **Problem:** Find the probability that the third head turns up in 5 tosses of an unbiased coin.
- This is a **negative binomial distribution** problem.
- Given:
  - $\boldsymbol{p = \frac{1}{2}}$ (success = head)
  - $\boldsymbol{r = 3}$ (3rd success)

Example 1: Setup

- **Problem:** Find the probability that the third head turns up in 5 tosses of an unbiased coin.
- This is a **negative binomial distribution** problem.
- Given:
  - $\boldsymbol{p = \frac{1}{2}}$ (success = head)
  - $\boldsymbol{r = 3}$ (3rd success)
  - $\boldsymbol{x + r = 5 \Rightarrow x = 2}$ (failures before 3rd success)

Example 1: Solution

- Using the negative binomial PMF:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

Example 1: Solution

- Using the negative binomial PMF:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

- Substituting values:

$$P(X = 2) = \binom{4}{2} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

Example 1: Solution

- Using the negative binomial PMF:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

- Substituting values:

$$P(X = 2) = \binom{4}{2} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

- Simplifying:

$$P(X = 2) = 6 \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{6}{32} = \frac{3}{16}$$

Example 2: Setup

- **Problem:** Find the probability that the third child in a family is the second daughter.

Example 2: Setup

- **Problem:** Find the probability that the third child in a family is the second daughter.
- Assumption: Male and female are equally likely $\Rightarrow \boldsymbol{p = \frac{1}{2}}$

Example 2: Setup

- **Problem:** Find the probability that the third child in a family is the second daughter.
- Assumption: Male and female are equally likely $\Rightarrow \boldsymbol{p = \frac{1}{2}}$
- This is again a **negative binomial distribution** problem.

Example 2: Setup

- **Problem:** Find the probability that the third child in a family is the second daughter.
- Assumption: Male and female are equally likely $\Rightarrow \boldsymbol{p = \frac{1}{2}}$
- This is again a **negative binomial distribution** problem.
- Given:
  - $\boldsymbol{p = \frac{1}{2}}$

Example 2: Setup

- **Problem:** Find the probability that the third child in a family is the second daughter.
- Assumption: Male and female are equally likely $\Rightarrow \boldsymbol{p = \frac{1}{2}}$
- This is again a **negative binomial distribution** problem.
- Given:
  - $\boldsymbol{p = \frac{1}{2}}$
  - $\boldsymbol{r = 2}$ (2nd daughter)

Example 2: Setup

- **Problem:** Find the probability that the third child in a family is the second daughter.
- Assumption: Male and female are equally likely $\Rightarrow p = \frac{1}{2}$
- This is again a **negative binomial distribution** problem.
- Given:
  - $p = \frac{1}{2}$
  - $r = 2$ (2nd daughter)
  - $x + r = 3 \Rightarrow x = 1$

Example 2: Solution

- Use the same negative binomial formula:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

Example 2: Solution

- Use the same negative binomial formula:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

- Substituting:

$$P(X = 1) = \binom{2}{1} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1$$

Example 2: Solution

- Use the same negative binomial formula:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

- Substituting:

$$P(X = 1) = \binom{2}{1} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1$$

- Simplifying:

$$P(X = 1) = 2 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{2}{8} = \frac{1}{4}$$

Example 3

- A proof-reader catches a misprint in a document with probability **0.8**.

Example 3

- A proof-reader catches a misprint in a document with probability **0.8**.
- Find the expected number of misprints in the document in which the proof-reader stops after catching the **20**$^{\text{th}}$ misprint.

Solution — Setup

- Let $\boldsymbol{X}$ be the number of misprints *not* caught by the proof-reader.

Solution — Setup

- Let $\boldsymbol{X}$ be the number of misprints *not* caught by the proof-reader.
- Let $\boldsymbol{r}$ be the number of misprints caught by the proof-reader. Here $\boldsymbol{r = 20}$.

Solution — Setup

- Let $\boldsymbol{X}$ be the number of misprints *not* caught by the proof-reader.
- Let $\boldsymbol{r}$ be the number of misprints caught by the proof-reader. Here $\boldsymbol{r = 20}$.
- The catch probability is $\boldsymbol{p = 0.8}$ so $\boldsymbol{q = 1 - p = 0.2}$.

Solution — Setup

- Let $\boldsymbol{X}$ be the number of misprints *not* caught by the proof-reader.
- Let $\boldsymbol{r}$ be the number of misprints caught by the proof-reader. Here $\boldsymbol{r = 20}$.
- The catch probability is $\boldsymbol{p = 0.8}$ so $\boldsymbol{q = 1 - p = 0.2}$.
- We model $\boldsymbol{X}$ by the **negative binomial** distribution (count of failures before $\boldsymbol{r}$ successes).

Solution — Mean of X

- For the negative binomial (failures before r successes), the mean is

$$\mathbb{E}(X) = \frac{rq}{p}.$$

Solution — Mean of X

- For the negative binomial (failures before r successes), the mean is

$$\mathbb{E}(X) = \frac{rq}{p}.$$

- Substitute $r = 20$, $q = 0.2$, $p = 0.8$:

$$\mathbb{E}(X) = \frac{20 \cdot 0.2}{0.8}.$$

Solution — Mean of X

- For the negative binomial (failures before r successes), the mean is

$$\mathbb{E}(X) = \frac{rq}{p}.$$

- Substitute $r = 20$, $q = 0.2$, $p = 0.8$:

$$\mathbb{E}(X) = \frac{20 \cdot 0.2}{0.8}.$$

- Compute:

$$\mathbb{E}(X) = \frac{4}{0.8} = 5.$$

Final Answer

- The expected total number of misprints when the proof-reader stops after catching the $\mathbf{20}^{\text{th}}$ is

$$\mathbb{E}(X + r) = \mathbb{E}(X) + r.$$

Final Answer

- The expected total number of misprints when the proof-reader stops after catching the $\mathbf{20}^{\text{th}}$ is

$$\mathbb{E}(\boldsymbol{X} + \boldsymbol{r}) = \mathbb{E}(\boldsymbol{X}) + \boldsymbol{r}.$$

- Using $\mathbb{E}(\boldsymbol{X}) = \mathbf{5}$ and $\boldsymbol{r} = \mathbf{20}$:

$$\mathbb{E}(\boldsymbol{X} + \boldsymbol{r}) = \mathbf{5} + \mathbf{20} = \mathbf{25}.$$

Final Answer

- The expected total number of misprints when the proof-reader stops after catching the $\mathbf{20}^{\text{th}}$ is
$$\mathbb{E}(\boldsymbol{X} + \boldsymbol{r}) = \mathbb{E}(\boldsymbol{X}) + \boldsymbol{r}.$$
- Using $\mathbb{E}(\boldsymbol{X}) = \mathbf{5}$ and $\boldsymbol{r} = \mathbf{20}$:
$$\mathbb{E}(\boldsymbol{X} + \boldsymbol{r}) = \mathbf{5} + \mathbf{20} = \mathbf{25}.$$
- **Hence, the expected number of misprints is 25.**

Example 4

- An item is produced by a machine in large numbers.

Example 4

- An item is produced by a machine in large numbers.
- The machine is known to produce **10%** defectives, i.e. $\boldsymbol{p = 0.1}$ and $\boldsymbol{q = 0.9}$.

Example 4

- An item is produced by a machine in large numbers.
- The machine is known to produce **10%** defectives, i.e. $\boldsymbol{p = 0.1}$ and $\boldsymbol{q = 0.9}$.
- A quality control engineer is testing the items randomly.

Example 4

- An item is produced by a machine in large numbers.
- The machine is known to produce **10%** defectives, i.e. $\boldsymbol{p = 0.1}$ and $\boldsymbol{q = 0.9}$.
- A quality control engineer is testing the items randomly.
- We are to find the probability that **at least 3 items are examined in order to get 2 defectives**.

Example 4

- An item is produced by a machine in large numbers.
- The machine is known to produce **10%** defectives, i.e. $\boldsymbol{p = 0.1}$ and $\boldsymbol{q = 0.9}$.
- A quality control engineer is testing the items randomly.
- We are to find the probability that **at least 3 items are examined in order to get 2 defectives**.
- This is a **negative binomial situation** where:

$$\boldsymbol{r = 2, \quad p = 0.1, \quad q = 0.9}$$

Solution

- Let X = number of non-defective items examined before getting $r = 2$ defectives.

Solution

- Let X = number of non-defective items examined before getting $r = 2$ defectives.
- We need $P(X + r \geq 3)$ or equivalently $P(X \geq 1)$.

Solution

- Let X = number of non-defective items examined before getting $r = 2$ defectives.
- We need $P(X + r \geq 3)$ or equivalently $P(X \geq 1)$.
- For the negative binomial distribution:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

Solution

- Let X = number of non-defective items examined before getting $r = 2$ defectives.
- We need $P(X + r \geq 3)$ or equivalently $P(X \geq 1)$.
- For the negative binomial distribution:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

- Thus,

$$P(X \geq 1) = 1 - P(X = 0)$$

Solution

- Let X = number of non-defective items examined before getting $r = 2$ defectives.
- We need $P(X + r \geq 3)$ or equivalently $P(X \geq 1)$.
- For the negative binomial distribution:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

- Thus,

$$P(X \geq 1) = 1 - P(X = 0)$$

- Now, $P(X = 0) = \binom{0+2-1}{2-1}(0.1)^2(0.9)^0 = 0.01$.

Solution

- Let X = number of non-defective items examined before getting $r = 2$ defectives.
- We need $P(X + r \geq 3)$ or equivalently $P(X \geq 1)$.
- For the negative binomial distribution:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

- Thus,

$$P(X \geq 1) = 1 - P(X = 0)$$

- Now, $P(X = 0) = \binom{0+2-1}{2-1}(0.1)^2(0.9)^0 = 0.01$.
- Therefore,

$$P(X \geq 1) = 1 - 0.01 = 0.99$$

Solution

- Let X = number of non-defective items examined before getting $r = 2$ defectives.
- We need $P(X + r \geq 3)$ or equivalently $P(X \geq 1)$.
- For the negative binomial distribution:

$$P(X = x) = \binom{x + r - 1}{r - 1} p^r q^x$$

- Thus,

$$P(X \geq 1) = 1 - P(X = 0)$$

- Now, $P(X = 0) = \binom{0+2-1}{2-1}(0.1)^2(0.9)^0 = 0.01$.
- Therefore,

$$P(X \geq 1) = 1 - 0.01 = 0.99$$

- Hence, the probability that at least 3 items are examined to get 2 defectives is $\boxed{0.99}$.

Example 5

- A family stops producing children after having the second daughter.

Example 5

- A family stops producing children after having the second daughter.
- Each child is equally likely to be male or female, i.e., $\boldsymbol{p = \frac{1}{2}}$.

Example 5

- A family stops producing children after having the second daughter.
- Each child is equally likely to be male or female, i.e., $\boldsymbol{p = \frac{1}{2}}$.
- Let $\boldsymbol{X}$ be the number of boys before the second girl is born.

Example 5

- A family stops producing children after having the second daughter.
- Each child is equally likely to be male or female, i.e., $\boldsymbol{p = \frac{1}{2}}$.
- Let $\boldsymbol{X}$ be the number of boys before the second girl is born.
- This is a **Negative Binomial Distribution** with:
  - $\boldsymbol{r = 2}$ (number of daughters),
  - $\boldsymbol{p = \frac{1}{2}}$ (probability of daughter),
  - $\boldsymbol{q = \frac{1}{2}}$ (probability of son).

Example 5

- A family stops producing children after having the second daughter.
- Each child is equally likely to be male or female, i.e., $p = \frac{1}{2}$.
- Let X be the number of boys before the second girl is born.
- This is a **Negative Binomial Distribution** with:
  - $r = 2$ (number of daughters),
  - $p = \frac{1}{2}$ (probability of daughter),
  - $q = \frac{1}{2}$ (probability of son).
- Expected number of boys:

$$E(X) = \frac{rq}{p} = \frac{2 \cdot \frac{1}{2}}{\frac{1}{2}} = 2$$

Example 5

- A family stops producing children after having the second daughter.
- Each child is equally likely to be male or female, i.e., $p = \frac{1}{2}$.
- Let X be the number of boys before the second girl is born.
- This is a **Negative Binomial Distribution** with:
  - $r = 2$ (number of daughters),
  - $p = \frac{1}{2}$ (probability of daughter),
  - $q = \frac{1}{2}$ (probability of son).
- Expected number of boys:

$$E(X) = \frac{rq}{p} = \frac{2 \cdot \frac{1}{2}}{\frac{1}{2}} = 2$$

- Total expected number of children:

$$E(X + r) = E(X) + r = 2 + 2 = 4$$

Applications

- Modeling number of failed trials before a fixed number of successes.

Applications

- Modeling number of failed trials before a fixed number of successes.
- Used in quality control (defective items before fixed number of good ones).

Applications

- Modeling number of failed trials before a fixed number of successes.
- Used in quality control (defective items before fixed number of good ones).
- Applied in ecology, insurance claims, and health studies.

Applications

- Modeling number of failed trials before a fixed number of successes.
- Used in quality control (defective items before fixed number of good ones).
- Applied in ecology, insurance claims, and health studies.
- Generalises the geometric distribution for multiple successes.

Reach PostNetwork Academy

Website

www.postnetwork.co

Reach PostNetwork Academy

Website

www.postnetwork.co

YouTube Channel

www.youtube.com/@postnetworkacademy

Reach PostNetwork Academy

Website

www.postnetwork.co

YouTube Channel

www.youtube.com/@postnetworkacademy

Facebook Page

www.facebook.com/postnetworkacademy

Reach PostNetwork Academy

Website

www.postnetwork.co

YouTube Channel

www.youtube.com/@postnetworkacademy

Facebook Page

www.facebook.com/postnetworkacademy

LinkedIn Page

www.linkedin.com/company/postnetworkacademy

Reach PostNetwork Academy

Website

www.postnetwork.co

YouTube Channel

www.youtube.com/@postnetworkacademy

Facebook Page

www.facebook.com/postnetworkacademy

LinkedIn Page

www.linkedin.com/company/postnetworkacademy

GitHub Repositories

www.github.com/postnetworkacademy

Thank You!