Set Theory Lecture Notes

BBC202 and NBCA204 - Discrete Mathematics

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Set Theory – Slide Overview (Part 1)

Set Theory – Slide Overview (Part 2)

Set Theory – Definition with Examples

Definition:
A set is a well-defined collection of distinct objects. The objects are called **elements** of the set.
If a belongs to A, we write $a \in A$. If a does not belong to A, we write $a \notin A$.

Examples:

1. Set of natural numbers: $A = \{1, 2, 3, 4, 5\}$
2. Set of vowels: $B = \{a, e, i, o, u\}$
3. Set of even numbers less than 10: $C = \{2, 4, 6, 8\}$
4. Set of days in a weekend: $D = \{Saturday, Sunday\}$
5. Empty set: $E = \emptyset$

Meaning of Well-Defined

Well-defined:
A set is said to be **well-defined** if there is no ambiguity in deciding whether an object belongs to the set or not.

Example (Well-defined):

- Set of even numbers less than 10: $\{2, 4, 6, 8\}$
- We can clearly decide membership.

Example (Not Well-defined):

- Set of beautiful flowers
- "Beautiful" is subjective, so membership is unclear.

Representation of a Set

A set can be represented in two main ways:

1. Roster or Tabular Form
2. Rule Method or Set-Builder Form

1. Roster or Tabular Form

Definition:
In this method, all elements of the set are listed inside curly braces { }.

Example 1:
Set of even numbers less than 10:

$$A = \{2, 4, 6, 8\}$$

Example 2:
Set of vowels:

$$B = \{a, e, i, o, u\}$$

Note: Elements are separated by commas and written only once.

2. Rule Method or Set-Builder Form

Definition:
In this method, a set is described by stating a rule or property that its elements satisfy.

General Form:

$$A = \{x \mid \text{x satisfies a given condition}\}$$

Example 1:
Set of even numbers less than 10:

$$A = \{x \mid x \text{ is even and } x < 10\}$$

Example 2:
Set of natural numbers less than 5:

$$B = \{x \mid x \in \mathbb{N}, x < 5\}$$

Finite and Infinite Sets

Finite Set:

$$A = \{1, 2, 3, 4\}$$

$$A = \{x \mid x \in \mathbb{N}, 1 \leq x \leq 4\}$$

Infinite Set:

$$B = \{1, 2, 3, \dots\}$$

$$B = \{x \mid x \in \mathbb{N}\}$$

Null (Empty) Set

A set with no elements is called a null set.

$$A = \emptyset$$

Set-Builder Form Example:

$$A = \{x \mid x \in \mathbb{N}, x < 0\}$$

Subset using Logical Implication

A set A is a subset of B if every element of A also belongs to B.

Logical Form:

$$A \subseteq B \iff (x \in A \Rightarrow x \in B)$$

Set-Builder Example:

$$A = \{x \mid x \in \mathbb{N}, 1 \leq x \leq 2\}$$

$$B = \{x \mid x \in \mathbb{N}, 1 \leq x \leq 4\}$$

Since any x satisfying $1 \leq x \leq 2$ also satisfies $1 \leq x \leq 4$,

$$A \subseteq B$$

Number of Subsets

If a set has n elements, number of subsets $= 2^n$.

$$A = \{1, 2\}$$

Set-Builder Form:

$$A = \{x \mid x \in \mathbb{N}, 1 \leq x \leq 2\}$$

Subsets:

$$\emptyset, \{1\}, \{2\}, \{1, 2\}$$

Superset and Proper Subset

Given Sets:

$$A = \{1, 2\}, \qquad B = \{1, 2, 3\}$$

Set-Builder Form:

$$A = \{x \mid x \in \mathbb{N}, 1 \leq x \leq 2\}$$

$$B = \{x \mid x \in \mathbb{N}, 1 \leq x \leq 3\}$$

Subset Check:
Every element of A (1,2) is present in B.

$$A \subset B$$

Proper Subset:
Since B contains an extra element (3), sets are not equal

Equal Set

Definition:

Two sets A and B are said to be **equal** if they contain exactly the same elements.

$$A = B \quad \text{if and only if} \quad (\forall x,\ x \in A \Leftrightarrow x \in B)$$

Example (Roster Form):

$$A = \{1, 2, 3\}, \qquad B = \{3, 2, 1\}$$

Since both sets contain the same elements,

$$A = B$$

Set-Builder Form:

$$A = \{x \mid x \in \mathbb{N}, 1 \leq x \leq 3\}$$

Universal Set

Definition:
A **Universal Set** is the set that contains all elements under consideration in a particular context. It is usually denoted by $\boldsymbol{U}$.

If $\boldsymbol{A}$ is any set in that context, then

$$\boldsymbol{A} \subseteq \boldsymbol{U}$$

Union of Sets

Definition:
The **union** of two sets A and B is the set of all elements that belong to A or B or both.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

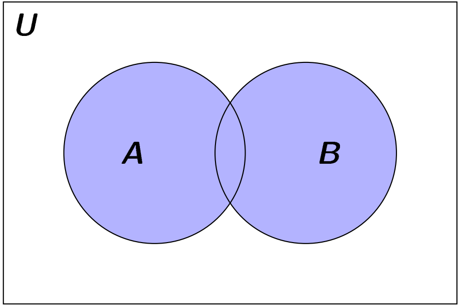
Example:

$$A = \{1, 2, 3\}, \quad B = \{3, 4, 5\}$$

$$A \cup B = \{1, 2, 3, 4, 5\}$$

Union of Sets

Venn Diagram (Universal Set U):



Intersection of Sets

Definition:
The **intersection** of two sets A and B is the set of common elements.

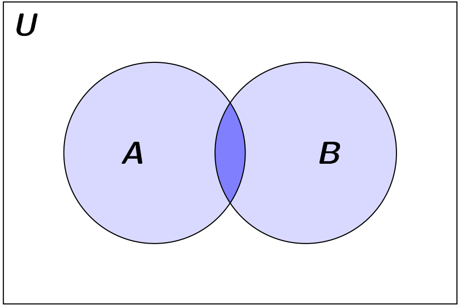
$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Example:

$$A = \{1, 2, 3\}, \quad B = \{3, 4, 5\}$$

$$A \cap B = \{3\}$$

Venn Diagram (Universal Set U):



Complement of a Set

Definition:
The **complement** of A is the set of elements in universal set U but not in A.

$$A' = U - A = \{x \mid x \in U \text{ and } x \notin A\}$$

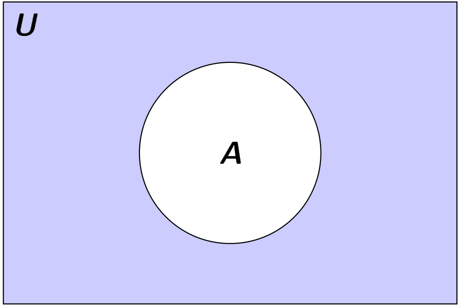
Example:

$$U = \{1, 2, 3, 4, 5\}, \quad A = \{1, 2, 3\}$$

$$A' = \{4, 5\}$$

Complement of a Set

Venn Diagram (Universal Set U):



Symmetric Difference

Definition:
The **symmetric difference** contains elements in A or B but not in both.

$$A \triangle B = (A - B) \cup (B - A)$$

$$A \triangle B = \{x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$$

Example:

$$A = \{1, 2, 3\}, \quad B = \{3, 4, 5\}$$

$$A \triangle B = \{1, 2, 4, 5\}$$

Venn Idea: Shade non-overlapping parts.

Symmetric Difference

Definition:

The **symmetric difference** contains elements in A or B but not in both.

$$A \triangle B = (A - B) \cup (B - A)$$

$$A \triangle B = \{x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$$

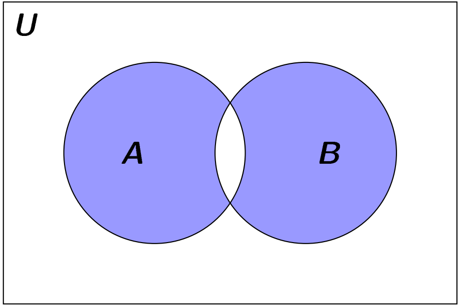
Example:

$$A = \{1, 2, 3\}, \quad B = \{3, 4, 5\}$$

$$A \triangle B = \{1, 2, 4, 5\}$$

Symmetric Difference

Venn Diagram (Universal Set U):



Idempotent Law

Statement:

$$A \cup A = A, \qquad A \cap A = A$$

Meaning: Repeating the same set does not change the set.

Example:

$$A = \{1, 2, 3\}$$

$$A \cup A = \{1, 2, 3\}, \qquad A \cap A = \{1, 2, 3\}$$

Set-Builder Proof:

$$A \cup A = \{x \mid x \in A \text{ or } x \in A\}$$

Since both conditions are same,

$$A \cup A = \{x \mid x \in A\} = A$$

Commutative Law

Statement:

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$

Meaning: Order of sets does not matter.

Example:

$$A = \{1, 2\},\ B = \{2, 3\}$$

$$A \cup B = \{1, 2, 3\} = B \cup A$$

Set-Builder Proof:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

This condition is same as

$$\{x \mid x \in B \text{ or } x \in A\}$$

Associative Law

Statement:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Meaning: Grouping does not matter.

Example:

$$A = \{1\}, B = \{2\}, C = \{3\}$$

$$(A \cup B) \cup C = \{1, 2, 3\} = A \cup (B \cup C)$$

Set-Builder Proof:

$$\{x \mid x \in A \text{ or } x \in B \text{ or } x \in C\}$$

Distributive Law

Statement:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Meaning: Intersection distributes over union.

Example:

$$A = \{1, 2\}, B = \{2, 3\}, C = \{2, 4\}$$

$$A \cap (B \cup C) = \{2\}$$

Set-Builder Proof:

$$\{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\}$$

Identity Law

Statement:

$$A \cup \emptyset = A, \qquad A \cap U = A$$

Meaning: Empty set adds nothing, universal set removes nothing.

Set-Builder Proof:

$$A \cup \emptyset = \{x \mid x \in A \text{ or } x \in \emptyset\}$$

Since no element belongs to $\emptyset$,

$$A \cup \emptyset = A$$

Practice Questions – Set Theory (1–10)

1. Define a set with two examples.
2. What is a well-defined set? Give one example.
3. Write two methods of representing a set.
4. Convert $A = \{2, 4, 6, 8\}$ into set-builder form.
5. Write roster form of $A = \{x \mid x \in \mathbb{N}, x < 5\}$.
6. Define finite and infinite sets with examples.
7. What is a null set? Give one example.
8. Define subset using logical implication.
9. If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, check $A \subseteq B$.
10. Find number of subsets of $A = \{a, b, c\}$.

Practice Questions – Set Theory (11–20)

11. Define proper subset with example.
12. Define superset with example.
13. When are two sets equal?
14. Define universal set with example.
15. Find $A \cup B$ if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.
16. Find $A \cap B$ for the above sets.
17. Define complement of a set with example.
18. If $U = \{1, 2, 3, 4, 5\}$ and $A = \{1, 2, 3\}$, find A'.
19. Define symmetric difference with example.
20. Find $A \triangle B$ if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$.

Practice Questions – Set Theory (21–25)

21. Verify Idempotent law for $A = \{1, 2\}$.
22. Verify Commutative law using $A = \{1, 2\}$ and $B = \{2, 3\}$.
23. Verify Associative law using $A = \{1\}, B = \{2\}, C = \{3\}$.
24. Verify Distributive law using suitable sets.
25. State and verify De Morgan's law with example.

Solutions – Set Theory (1–5)

1. Define a set with two examples

A set is a well-defined collection of distinct objects.

Examples: $\{1, 2, 3\}$, $\{a, e, i, o, u\}$

2. Well-defined set

A set where membership is clear.

Example: $\{2, 4, 6, 8\}$

3. Methods of representation

Roster form and Set-builder form

4. Convert to set-builder

$A = \{2, 4, 6, 8\}$

$A = \{x \mid x \text{ is even and } x < 10\}$

5. Roster form

$A = \{x \mid x \in \mathbb{N}, x < 5\} = \{1, 2, 3, 4\}$

Solutions – Set Theory (6–10)

6. Finite and Infinite sets

Finite: $\{1, 2, 3\}$ Infinite: $\{1, 2, 3, \dots\}$

7. Null set

Set with no elements. Example: $\{x \mid x < 0, x \in \mathbb{N}\}$

8. Subset using implication

$$A \subseteq B \iff (x \in A \Rightarrow x \in B)$$

9. Check subset

$A = \{1, 2\}, B = \{1, 2, 3\}$

All elements of A are in $B \Rightarrow A \subseteq B$

10. Number of subsets

$A = \{a, b, c\}$

$2^3 = 8$ subsets

Solutions – Set Theory (11–15)

11. Proper subset
$A \subset B$ and $A \neq B$
Example: $\{1, 2\} \subset \{1, 2, 3\}$

12. Superset
$B \supset A$
Example: $\{1, 2, 3\} \supset \{1, 2\}$

13. Equal sets
$A = B$ if both have same elements

14. Universal set
Set containing all elements. Example: $U = \{1, 2, 3, 4, 5\}$

15. Union
$A \cup B = \{1, 2, 3, 4, 5\}$

16. Intersection
$A \cap B = \{3\}$

17. Complement
$A' = \{x \mid x \in U, x \notin A\}$

18. Find complement
$U = \{1, 2, 3, 4, 5\}, A = \{1, 2, 3\}$
$A' = \{4, 5\}$

19. Symmetric difference
$A \triangle B = (A - B) \cup (B - A)$

20. Find symmetric difference
$A \triangle B = \{1, 2, 4, 5\}$

Solutions – Set Theory (21–25)

21. Idempotent law

$A = \{1, 2\}$

$A \cup A = A$, $A \cap A = A$

22. Commutative law

$A \cup B = \{1, 2, 3\} = B \cup A$

23. Associative law

$(A \cup B) \cup C = \{1, 2, 3\} = A \cup (B \cup C)$

24. Distributive law

$A = \{1, 2\}, B = \{2, 3\}, C = \{2, 4\}$

$A \cap (B \cup C) = \{2\} = (A \cap B) \cup (A \cap C)$

25. De Morgan's law

$(A \cup B)' = A' \cap B'$